# Formulae you need to know

## Binomial Theorem

Factorials: $n! = n \times (n-1) \times (n-2) \times \ldots\ldots.. \times 2 \times 1 \qquad 1! = 1 \qquad 0! = 1$

Permutations: ${}^{n}P_{r} = \dfrac{n!}{r!}$

Number of ways of **arranging** *r* objects chosen from a set of *n*.

e.g. How many ways can the first 3 places be allocated for a race with 7 runners

Combinations: ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$

Number of ways of **choosing** *r* objects chosen from a set of *n* where the order of choice

e.g. How many ways are there of choosing 3 colours from a palette of 7

Compact Notation: $\dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Factorial Relationships: $\dbinom{n}{r} = \dbinom{n}{n-r} \quad \textit{and} \quad \dbinom{n}{0} = \dbinom{n}{n} = 1 \quad \textit{and} \quad \dbinom{n}{1} = \dbinom{n}{n-1} = n$

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

Binomial Theorem: $(x+y)^n = \displaystyle\sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \qquad$ General term: $\dbinom{n}{r} x^{n-r} y^r$

i.e $(x+y)^n = \dbinom{n}{0} x^n + \dbinom{n}{1} x^{n-1} y + \dbinom{n}{2} x^{n-2} y^2 + \ldots\ldots\ldots + \dbinom{n}{n} y^n$

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## Partial Fractions

### Proper rational fractions

where denominator contains:

i) Distinct linear factors: $\dfrac{3}{(x+1)(x-3)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x-3}$

ii) Repeated Linear Factor: $\dfrac{3}{(x+1)^2(x-3)} \equiv \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-3}$

iii) Irreducible quadratic Factor: $\dfrac{3}{(x+1)(x^2+2x+2)} \equiv \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2x+2}$

### Improper rational fractions

Use algebraic division to extract whole part and the remainder which is a proper rational fraction.

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**Differentiation 1**

In the following formulae, (**Leibniz form**), $u$ represents $u(x)$ and $v$ represents $v(x)$

Limit formula: $f'(x) = \lim_{h \to 0} \left( \frac{f(x+h) - f(x)}{h} \right)$

Chain Rule: $k(x) = f(g(x)) \quad \rightarrow \quad k'(x) = f'(g(x)) \cdot g'(x)$

Leibniz: $y = u(v) \quad \rightarrow \frac{dy}{dx} = u'(v) \cdot v'$

Product Rule: $k(x) = f(x) \cdot g(x) \quad \rightarrow \quad k'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$

Leibniz: $y = u \cdot v \quad \rightarrow \frac{dy}{dx} = u \cdot v' + u' \cdot v$

Quotient Rule: $k(x) = \frac{f(x)}{g(x)} \quad \rightarrow \quad k'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$

Leibniz: $y = \frac{u}{v} \quad \rightarrow \frac{dy}{dx} = \frac{u' \cdot v - u \cdot v'}{v^2}$

**Standard Derivatives:**

You should be able to work these out if required.

Use the definitions: $\tan x = \frac{\sin x}{\cos x}$; $\sec x = \frac{1}{\cos x}$; $\operatorname{cosec} x = \frac{1}{\sin x}$; $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

and use chain rule or quotient rule as appropriate.

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x \qquad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

Exponential and logarithmic functions:

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Useful tricks: $e^{\ln x} = x$ and $\ln e^x = x$

Change of base: $\log_a x = \frac{\log_b x}{\log_b a} \quad \rightarrow \quad \log_a x = \frac{\log_{10} x}{\log_{10} a} \quad \rightarrow \quad \log_a x = \frac{\ln x}{\ln a}$

*This is useful for evaluating and for differentiating.*

Higher Derivatives: $\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

## Applications of Differentiation 1

### Rectilinear Motion

Displacement: $x = x(t)$

Velocity: $v = \dfrac{dx}{dt} = x'(t)$

Acceleration: $a = \dfrac{dv}{dt} = v'(t) = \dfrac{d^2x}{dt^2} = x''(t)$

Time *t* is the common value that is true in all the equations. You can find the displacement, velocity and acceleration from one particular value of *t*. You can also reverse these using integration.

Also consider initial values - when *t* = 0 this gives the initial conditions for displacement, velocity and acceleration.

Maximum height occurs when velocity = 0. Find *t* for when this occurs.

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### Extrema and critical points

Extrema: The extreme values of a function
e.g. global maxima, minima, local maxima, minima

Critical points: Any point $(a, f(a))$
where $f'(a) = 0$ or where $f'(a)$ does not exist

Stationary Points: $f'(x) = 0$

Table of signs

Second derivative test: $\dfrac{d^2y}{dx^2} > 0$ minimum

$\dfrac{d^2y}{dx^2} < 0$ maximum

$\dfrac{d^2y}{dx^2} = 0$ inconclusive – draw a table of signs.

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### Newton's Method of approximating roots

Using trial by improvement to solve $f(x) = 0$ $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Take a guess. If $x_n$ is your guess, then $x_{n+1}$ is a better guess.

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### Optimisation Problems

As Higher, but may involve newly learned differentiation techniques.

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**Integration 1**

**Standard Forms**

$$\int e^x \, dx = e^x + C$$

$$\int \frac{1}{x} \, dx = \ln x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

**Integration by Substitution**

Find the essential function and reduce the integral to a standard form, that we know how to do.
If using the substitution: $u = f(x)$ then you have to change the differential $dx$ to $du$

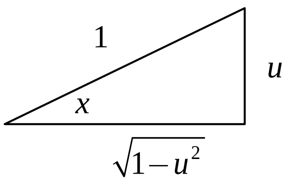
by using the derivative of $f(x)$ i.e. $u = f(x) \rightarrow \dfrac{du}{dx} = f'(x) \rightarrow dx = \dfrac{1}{f'(x)} du$

and remove any remaining functions of $x$ by using the relationship $u = f(x)$ in some form.

**Useful tricks**

When using trigonometric substitutions other ratios can be found from a right angled triangle and using Pythagoras.

e.g. $u = \sin x$




hence: $\cos x = \sqrt{1-u^2}$ etc.

also $\displaystyle\int \sin^5 x \, dx = \int \left(\sin^2 x\right)^2 \sin x \, dx = \int \left(1-\cos^2 x\right)^2 \sin x \, dx$ then use $u = \cos x$

**Substitution and Definite Integrals**

Change the limits of integration using the substitution to evaluate the integral using the substituted variable.

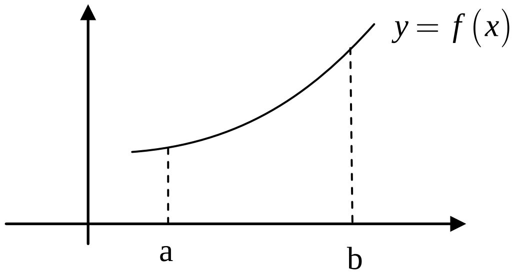
**Special or common forms**

$$\int \frac{f'(x)}{f(x)} \, dx = \ln\left|f(x)\right| + C$$

$$\int f'(x) \cdot f(x) \, dx = \frac{1}{2}\left(f(x)\right)^2 + C \quad \text{(result of the chain rule)}$$

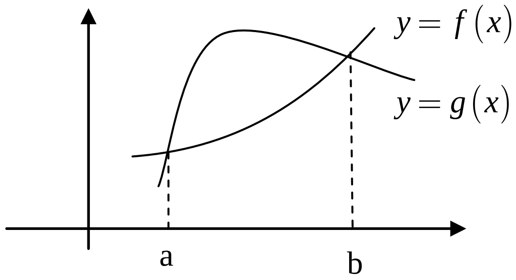
## Area under a curve

Area between curve and the *x*-axis.



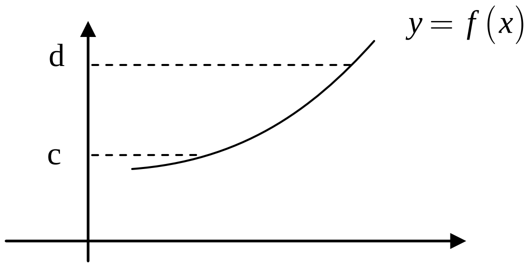

$$\int_a^b f(x)\,dx$$

Area between two curves




$$\int_a^b g(x) - f(x)\,dx$$

Area between curve and the *y*-axis.



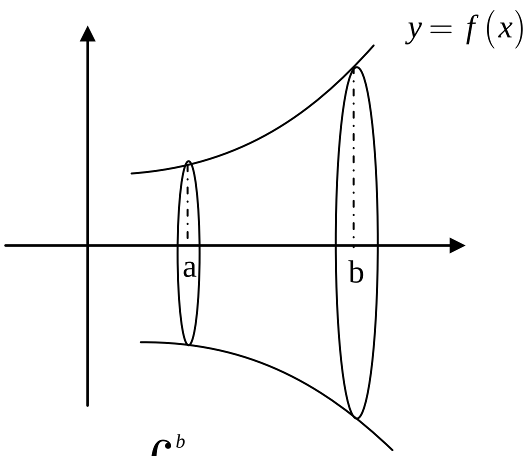

$$\int_c^d g(y)\,dy$$

**NB:** This can only be done, if you can express $y = f(x)$ in the form $x = g(y)$.

An **alternative method** is to find area between the curve and the x-axis and subtract it from a suitable rectangle with appropriate coordinates.

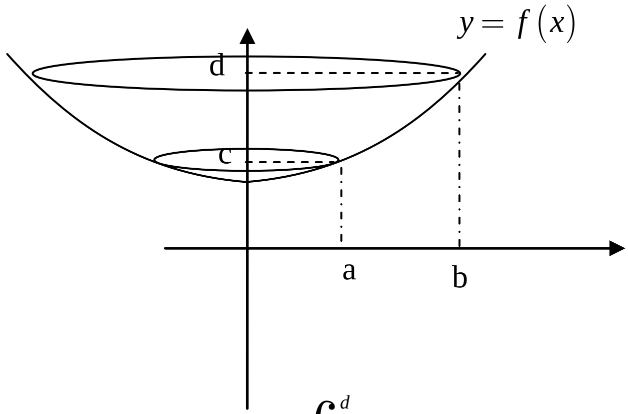
## Volumes of revolution

Rotate $y = f(x)$ around *x*-axis.




$$\int_a^b \pi y^2\ dx$$

Rotate $y = f(x)$ around *y*-axis.




$$\int_c^d \pi x^2\ dx$$

These become difficult to evaluate except for simple functions.

## Properties of functions

The modulus Function: $y = |x|$ for the graph **reflect** any **negative** portions in the *x*-axis.

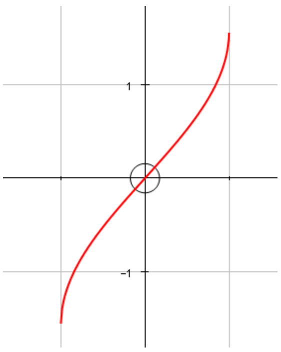
Inverse functions: Reflect the graph in the line.

$y = f(x) \iff x = f^{-1}(y)$

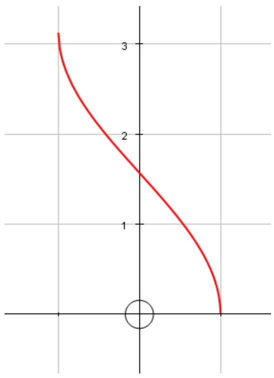
Can sometimes, but not always change the subject of the formula.

For function to have an inverse it must be increasing or decreasing over all of its domain.

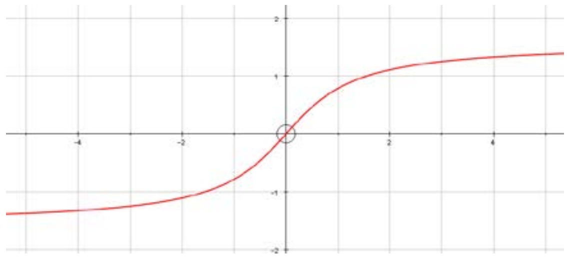
Know graphs of inverse trig functions:



$y = \sin^{-1} x$



$y = \cos^{-1} x$



$y = \tan^{-1} x$

## Sketching Polynomials

Intersections with *x* and *y* axes.
Stationary points
Behaviour of *y* as $x \to \infty$

## Concavity and Points of Inflection

Concave up – where the gradient is increasing
Concave down – where the gradient is decreasing

## Odd and even functions

$f(-x) = f(x)$ graph is symmetrical about *y*-axis. EVEN function.

$f(-x) = -f(x)$ graph is has half-turn symmetry about the origin. ODD function.

## Vertical, Horizontal and Oblique asymptotes

Vertical asymptotes: $f(x) = \dfrac{1}{x-1}$ x = 1 is a vertical asymptote (denominator = 0)

Horizontal asymptotes: $f(x) = g(x) + \dfrac{m(x)}{n(x)}$ where $\dfrac{m(x)}{n(x)}$ is a proper rational function

If $g(x)$ is a linear function,

then $y = g(x)$ is a horizontal or oblique asymptote depending on gradient of $g(x)$

**Graphs of related functions**

Given the graph of $y = f(x)$ then the related graph:

| | |
|---|---|
| $y = -f(x)$ | is a reflection in the *x*-axis. |
| $y = f(-x)$ | is a reflection in the *y*-axis. |
| $y = f(x+a)$ | is a translation *a* units to the left. |
| $y = f(x-a)$ | is a translation *a* units to the right. |
| $y = f(x)+a$ | is a translation *a* units up. |
| $y = f(x)-a$ | is a translation *a* units down. |
| $y = f(kx)$ | is a scaling in the *x*-direction. *x*-stretch if $0 < k < 1$; *x*-squash if $k > 1$ |
| $y = kf(x)$ | is a scaling in the *y*-direction. *y*-stretch if $k > 1$; *y*-squash if $0 < k < 1$ |

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**Systems of Equations**

Matrix form:

$$\begin{aligned} x+2y+3z&=6 \\ x-3y+4z&=2 \\ 2y-5z&=1 \end{aligned} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 1 & -3 & 4 \\ 0 & 2 & -5 \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & -3 & 4 & 2 \\ 0 & 2 & -5 & 1 \end{array}\right)$$

The augmented matrix

Elementary Row Operations:

Switching the rows
Add or subtract one row to or from another
Multiply or divide a row by a constant.

Upper triangular form $\left(\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 0 & -8 & 6 & 4 \\ 0 & 0 & -1 & 1 \end{array}\right)$ (NB this is an arbitrary example)

Solve the equations by back substitution or by reducing the matrix further to 1s on the leading diagonal mean that you can read the solution directly from the matrix.

e.g. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$ gives $x = 2$; $y = 1$; $z = -1$

Redundancy and Inconsistency:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

A row of zeros indicate redundancy

All zeros on the left and a number on the right indicates inconsistency.

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**Further Differentiation**

**Derivatives of Inverse Functions**

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)} \qquad\qquad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

**Standard derivatives**

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

**Implicit Differentiation**

e.g. $x^3 - 2xy^2 + y^3 = 0$ Differentiating $3x^2 - 2x\cdot 2y\frac{dy}{dx} - 2y^2 + 3y^2\frac{dy}{dx} = 0$

Re-arrange: $\left(3y^2 - 4xy\right)\frac{dy}{dx} = 2y^2 - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{2y^2 - 3x^2}{3y^2 - 4xy}$

**Logarithmic Differentiation**

Useful when function is complicated by powers, roots, products and quotients of several factors or where the variable appears in an index.

Take logs of both sides before differentiating – and then differentiate implicitly.

e.g. $y = x^x \rightarrow \ln y = \ln x^x \rightarrow \ln y = x\ln x$

differentiating: $\rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + 1\cdot\ln x \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = 1 + \ln x \Rightarrow \frac{dy}{dx} = \left(1+\ln x\right)x^x$

**Parametric Differentiation**

i) You can use the constraint equation which is formed by eliminating the parameter, and then use implicit differentiation.

ii) Differentiate each parametric equation separately and use the chain rule technique to join them together.

If $x = x(t); \quad y = y(t)$ then $\frac{dx}{dt} = x'(t); \quad \frac{dy}{dt} = y'(t)$

hence: $f'(x) = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = y'(t)\cdot\frac{1}{x'(t)} = \frac{y'(t)}{x'(t)}$

also: $f''(x) = \frac{x'(t)y''(t) - x''(t)y'(t)}{\left(x'(t)\right)^3}$

**Related Rates of Change:**

When one variable $x$, is a function of another, $u$, then $\frac{dx}{du} = \frac{1}{\frac{du}{dx}}$

e.g. $\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{dt}{dV}$

Spheres: Volume: $V = \frac{4}{3}\pi r^3$ Surface Area: $A = 4\pi r^2$

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**Further Integration**

Standard integrals

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

**Integrals of rational functions**

Try to split up the function using partial fractions

$$\int \frac{1}{a+bx}\,dx = \frac{1}{b}\ln\left|a+bx\right| + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + C$$

**Integration by parts**

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \qquad \text{Rearrange:} \quad \Rightarrow \quad \frac{d}{dx}(u \cdot v) - v \cdot \frac{du}{dx} = u \cdot \frac{dv}{dx}$$

i.e. $u \cdot \frac{dv}{dx} = \frac{d}{dx}(u \cdot v) - v \cdot \frac{du}{dx}$

and integrate $\rightarrow \int u \cdot \frac{dv}{dx} = \int \frac{d}{dx}(u \cdot v) - \int v \cdot \frac{du}{dx}$

Hence: $\rightarrow \int u \cdot \frac{dv}{dx} = u \cdot v - \int v \cdot \frac{du}{dx}$

Choose for *v* the function you can integrate and *u* for the function you can differentiate.

Sometimes you have to integrate by parts repeatedly, and on some occasions you return to where you start – often with sine and cosine functions.

Sometimes you choose a 'dummy function' eg. $\frac{dv}{dx}$ = 1, when you cannot integrate directly.

$$\text{eg.} \quad \int \ln x \; dx \;\rightarrow\; \int 1 \cdot \ln x \; dx \;\rightarrow\; u = \ln x \quad \frac{dv}{dx} = 1$$

**Differential equations**

First order, first degree

Integrate by separating the variables.

e.g. $\frac{dy}{dx} = x^3 + 4x$ $\qquad y = \int x^3 + 4x \, dx$

$\frac{dy}{dx} = f(y)$ $\qquad \frac{1}{f(y)} \, dy = dx \;\Rightarrow\; \int \frac{1}{f(y)} \, dy = \int 1 \, dx$

Variables separable

e.g. $\frac{dy}{dx} = f(x) \cdot g(y)$ $\qquad \frac{1}{g(y)} \, dy = f(x) \, dx \;\Rightarrow\; \int \frac{1}{g(y)} \, dy = \int f(x) \, dx$

**Applications:**

Form the differential equation from the information in the question and then solve it using the above techniques.

You can find the Particular Solution by evaluating the constant of integration, C, using the given conditions in the question.